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Jayawardhana, B.; Weiss, G.

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Convergence of the state of a passive nonlinear plant with an L^2 input

Bayu Jayawardhana and George Weiss

Abstract—In this paper, we consider a strictly output passive nonlinear plant \mathbf{P} with storage function H . We assume that \mathbf{P} is zero-state detectable. Under some mild conditions on H , we show that the state x of the plant converges to zero for any L^2 input. This implies the solvability for all $t \geq 0$ of the system equations, for every input in L^2_{loc} .

We define a stability notion called L^2 system-stable, a variant to the L^2 -stability concept, which has a nice interconnection properties.

I. INTRODUCTION

Passive systems have a \mathcal{C}^1 storage function H (defined on the state space \mathbb{R}^n) which has the intuitive meaning of stored energy. The input signal u and the output signal y take values in the same inner product space. We denote the state of the system at time t by $x(t)$. The defining property of a passive system is that if a state trajectory exists then

$$\dot{H} \leq \langle y, u \rangle, \quad \text{where} \quad \dot{H} = \frac{\partial H(x)}{\partial x} \dot{x}. \quad (1)$$

The dynamics of many physical systems such as electrical circuits or mechanical systems can be described as passive systems, if one chooses properly the input u and the output y . The product of y and u should correspond to the power flow into the system.

It is known that passive systems have inherent stability properties. The Lyapunov stability of the equilibrium points corresponding to $u = 0$ can be shown by using H as a Lyapunov function (see, for example, Willems [19]). A stability property that some passive systems have is L^2 -stability, i.e., if the input u is in L^2 (for $t \geq 0$), then (for any initial state) the equations of the system have a unique solution (for all $t \geq 0$) and the output y is also in L^2 (see van der Schaft [14] for details).

It is shown in [14] that a strictly output passive system, i.e., a passive system where the storage function H satisfies

$$\dot{H} \leq \langle y, u \rangle - k\|y\|^2, \quad k > 0, \quad (2)$$

has an L^2 gain $\leq \frac{1}{k}$. Such a system is locally asymptotically stable at 0 if it is zero-state detectable [14]. Moreover, if $\lim_{\|x\| \rightarrow \infty} H(x) = \infty$ (i.e., H is proper) then the system is globally asymptotically stable at 0.

Many references study the conditions under which a nonlinear system is passive, and when a nonlinear system can be

made passive by state feedback. For affine nonlinear systems, Moylan [12] described necessary and sufficient conditions for the system to be passive. The conditions are analogous to the Kalman-Yakubovich-Popov conditions for linear time-invariant systems. In Byrnes *et al* [1], it is shown that if a nonlinear system has relative degree one and it is minimum phase, then the system can be rendered passive by state feedback.

Based on these results, passivity-based controller design exploits the stability properties of passive systems. Ortega *et al* [13] describes several passivity-based controller design methods for electrical and mechanical systems modeled by Euler-Lagrange equations. The book [14] introduced passivity-based control for port-controlled Hamiltonian systems. Lozano *et al* [11] describes control applications of dissipative systems theory. Jayawardhana [3] proposes a controller design to reject input disturbance signals generated by an exosystem and to track constant reference signals. For a fully actuated mechanical system, a passivity-based tracking controller has been proposed by Slotine and Li [15] (see also [13] for the passivity property of the closed-loop system using the Slotine-Li controller). The combination of the Slotine-Li controller with an internal model is explored in our paper [4].

In this paper, we study the behavior of a strictly output passive nonlinear system given an L^2 input signal. Under mild assumptions on the differential equation, we show that if the system is zero-state detectable and its storage function H is proper (these concepts are defined in Section II), then the state x converges to zero. This implies that a unique solution of the differential equation exists for all $t \geq 0$, for any input signal in L^2_{loc} .

The intuition behind our main result is the following: According to the global asymptotic stability result stated after (2), when $u = 0$ then $x(t) \rightarrow 0$. If $u \in L^2$, then for very large τ the energy left in u for $t \geq \tau$ becomes negligible, and the system behaves as it would for $u = 0$, i.e., we have $x(t) \rightarrow 0$. However, a rigorous proof of this result is not easy. Our proof uses techniques from infinite-dimensional system theory.

The main result of this paper has been used in our paper [5] to solve an input disturbance rejection problem, where the disturbance can be decomposed into a signal generated by an exosystem and an L^2 signal. The technique used in this paper can also be used for certain nonlinear systems to show the convergence of the state given an L^p input signal, where $p \in [1, \infty)$, see Jayawardhana [6].

The linear version of our main result is the following: For a linear time-invariant (LTI) system \mathbf{P} which is detectable and strictly output passive we have $x(t) \rightarrow 0$, for every L^2

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The first author is with the Dept. Mathematical Sciences, University of Bath, Bath BA2 7AY. The second author is with the Control and Power Group, Dept. of Electrical and Electronic Engineering, Imperial College London, London SW7 2AZ, UK. e-mail: bayujw@ieee.org, g.weiss@imperial.ac.uk

input u . The proof of this is easy: suppose that \mathbf{P} is described by

$$\dot{x} = Ax + Bu, \quad y = Cx + Du, \quad (3)$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$. From the detectability and the strict output passivity of \mathbf{P} , it follows that \mathbf{P} is stable. Thus, $u \in L^2$ implies that $x \in L^2$. From (3), we also have that $\dot{x} \in L^2$. Using Barbălat's lemma (see Logemann and Ryan [10]), it follows that $x(t) \rightarrow 0$.

II. PRELIMINARIES

Notation. Throughout this paper, the inner product on any Hilbert space is denoted by $\langle \cdot, \cdot \rangle$ and $\mathbb{R}_+ = [0, \infty)$. We refer to [8] and [14] for basic concepts on nonlinear systems and on passivity theory. For a finite-dimensional vector x , we use the norm $\|x\| = (\sum_n |x_n|^2)^{\frac{1}{2}}$ and for matrices, we use the operator norm induced by $\|\cdot\|$ (the largest singular value). For any $\varepsilon \geq 0$, we denote $\mathbf{B}_\varepsilon = \{x \in \mathbb{R}^n \mid \|x\| \leq \varepsilon\}$. For any finite-dimensional vector space \mathcal{V} endowed with a norm $\|\cdot\|_{\mathcal{V}}$, the space $L^2(\mathbb{R}_+, \mathcal{V})$ consists of all the measurable functions $f: \mathbb{R}_+ \rightarrow \mathcal{V}$ such that $\int_0^\infty \|f(t)\|_{\mathcal{V}}^2 dt < \infty$. The square-root of the last integral is denoted by $\|f\|_{L^2}$. For $f \in L^2(\mathbb{R}_+, \mathcal{V})$ and $T > 0$, we denote by f_T the truncation of f to $[0, T]$. The space $L_{loc}^2(\mathbb{R}_+, \mathcal{V})$ consists of all the measurable functions $f: \mathbb{R}_+ \rightarrow \mathcal{V}$ such that $f_T \in L^2(\mathbb{R}_+, \mathcal{V})$, for all $T > 0$. The space $\mathcal{H}^1(\mathbb{R}_+, \mathcal{V})$ consists of all the functions $f: \mathbb{R}_+ \rightarrow \mathcal{V}$ such that $f, \frac{df}{dt} \in L^2(\mathbb{R}_+, \mathcal{V})$ (where $\frac{df}{dt}$ is understood in the sense of distributions). The space $\mathcal{C}(\mathbb{R}^l, \mathbb{R}^p)$ (respectively $\mathcal{C}^1(\mathbb{R}^l, \mathbb{R}^p)$) consists of all the continuous (respectively continuously differentiable) functions $f: \mathbb{R}^l \rightarrow \mathbb{R}^p$.

Consider the time-invariant plant \mathbf{P} described by

$$\dot{x} = f(x, u), \quad (4)$$

$$y = h(x), \quad (5)$$

where the state x , the input u and the output y are functions of $t \geq 0$, such that $x(t) \in \mathbb{R}^n$, $u(t), y(t) \in \mathbb{R}^m, m \leq n$. We assume that $f \in \mathcal{C}^1(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n)$ with $f(0, u) = 0 \Leftrightarrow u = 0$ and $h \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^m)$ with $h(0) = 0$. We assume that there exists a storage function $H \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}_+)$ such that for some $k > 0$,

$$\frac{\partial H(x)}{\partial x} f(x, u) \leq \langle y, u \rangle - k\|y\|^2. \quad (6)$$

The plant \mathbf{P} as in (4)–(5) with the storage function H satisfying (6) is strictly output passive, which means that it satisfies (2) (this is easy to verify). H is called *proper* if $H(x) \rightarrow \infty$ whenever $\|x\| \rightarrow \infty$.

We recall a result on the existence and uniqueness of the solution of the differential equation (4) (see also Sontag [16, Appendix C] for details).

Definition 2.1: A solution of (4) with a measurable input u on an interval \mathcal{I} containing 0 is an absolutely continuous function $x: \mathcal{I} \rightarrow \mathbb{R}^n$ such that

$$x(t) - x(0) = \int_0^t f(x(\tau), u(\tau)) d\tau \quad \forall t \in \mathcal{I}.$$

Theorem 2.2: Assume that $u: \mathbb{R}_+ \rightarrow \mathbb{R}^m$ is measurable, $f \in \mathcal{C}(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n)$ and the following two conditions hold for every $a \in \mathbb{R}^n$:

- (S1) There exists a constant $c > 0$ and a locally integrable function $\alpha: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that

$$\|f(x, u(t)) - f(y, u(t))\| \leq \alpha(t)\|x - y\|$$

for almost every $t \in \mathbb{R}_+$ and for all $x, y \in a + \mathbf{B}_c$.

- (S2) There exists a locally integrable function $\beta: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that for almost every $t \in \mathbb{R}_+$,

$$\|f(a, u(t))\| \leq \beta(t).$$

Then for every $x(0) \in \mathbb{R}^n$ there exists $\delta > 0$ and a unique solution of (4) with input u on $[0, \delta)$.

This theorem is an immediate consequence of Theorem 36 in [16]. We need this result in Section 3 when dealing with an L^2 input signal.

Corollary 2.3: Suppose that u and f are as in Theorem 2.2 and for some $x(0) \in \mathbb{R}^n$, $[0, \delta)$ (where $\delta > 0$) is the maximal interval of existence of the solution of (4). If $\delta < \infty$ then for every compact set $\mathcal{K} \subset \mathbb{R}^n$, there exists $T \in [0, \delta)$ such that $x(T) \notin \mathcal{K}$.

Proof: The property (S1) and (S2) in Theorem (2.2) implies also that for any compact $K \subset \mathbb{R}^n$, there is a locally integrable function γ such that

$$\|f(x, u(t))\| \leq \gamma(t), \quad (7)$$

for almost every $t \in \mathbb{R}_+$ and for all $x \in \mathcal{K}$. Indeed, given any $a \in K$, there exists $c > 0$ and function α and β as in the Theorem 2.2. Thus,

$$\|f(x, t)\| \leq \|f(a, t)\| + \|f(x, t) - f(a, t)\| \leq \beta(t) + c\alpha(t),$$

for all $x \in a + \mathbf{B}_c$ and almost every $t \in \mathbb{R}_+$. Denote the last inequality above by $\gamma_a(t) = \beta(t) + c\alpha(t)$ which is locally integrable. Consider the open covering of K by the sets of the form $\mathbf{B}_{c_j} + a_j$, $a_j \in K$, $j = \{1, 2, \dots\}$. By compactness, the open covering has a finite subcovering, i.e., j is finite. Choose $\gamma(t) = \max_j \{\gamma_{a_j}(t)\}$, then γ satisfies (7) since γ_j is locally integrable for each j .

We prove the corollary by using contradiction. Suppose that there exists a compact set $K \subset \mathbb{R}^n$ such that $x(t) \in K$ for all $t \in [0, I(x(0)))$. First, we show that $\lim_{t \rightarrow I(x(0))} x(t)$ exists. For the compact set K , we know that there exists a locally integrable function $\gamma: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that (7) holds. Then we have

$$\|x(t_k) - x(t_j)\| \leq \int_{t_j}^{t_k} \|f(x(\tau), u(\tau))\| d\tau \leq \int_{t_j}^{t_k} \gamma(\tau) d\tau,$$

where $t_k, t_j \in [0, I(x(0)))$. Since $\|x(t_k) - x(t_j)\| \rightarrow 0$ as $t_k, t_j \rightarrow I(x(0))$. Since K is a complete metric space, $\lim_{t \rightarrow I(x(0))} x(t)$ exists and $x(I(x(0))) \in K$. However, we could use again Theorem 2.2 with $I(x(0))$ as the initial time and $x(I(x(0)))$ as the initial state to show the existence of solution of (4) on the interval $[I(x(0)), \eta)$, $\eta > I(x(0))$. This shows that $I(x(0))$

is not the maximal interval of existence of the solution of (4). \square

If $I(x(0)) < \infty$ is as in Corollary 2.3, then it is called *the finite escape time*.

Let \mathcal{X} be a metric space with distance μ . A set $G \subset \mathcal{X}$ is *relatively compact* if the closure of G is compact. Let $z: \mathbb{R}_+ \rightarrow \mathcal{X}$. A point $\xi \in \mathcal{X}$ is said to be an ω -limit point of z if there exists a sequence (t_n) in \mathbb{R}_+ such that $t_n \rightarrow \infty$ and $z(t_n) \rightarrow \xi$. The set of all the ω -limit points of z is denoted by $\Omega(z)$.

A map $\pi: \mathbb{R}_+ \times \mathcal{X} \rightarrow \mathcal{X}$ is said to be a *semiflow on \mathcal{X}* if π is continuous, $\pi(0, x_0) = x_0$ for all $x_0 \in \mathcal{X}$ and

$$\pi(s+t, x_0) = \pi(s, \pi(t, x_0)) \quad \forall s, t \in \mathbb{R}_+ \quad \forall x_0 \in \mathcal{X}.$$

A non-empty set $G \subset \mathcal{X}$ is π -invariant if $\pi(t, G) = G$ for all $t \in \mathbb{R}_+$.

Proposition 2.4: Let $\pi: \mathbb{R}_+ \times \mathcal{X} \rightarrow \mathcal{X}$ be a semiflow on a metric space \mathcal{X} . Let $x_0 \in \mathcal{X}$ and denote $z(t) = \pi(t, x_0)$. If $z(\mathbb{R}_+)$ is relatively compact, then $\Omega(z)$ is non-empty, compact, π -invariant and

$$\lim_{t \rightarrow \infty} \mu(z(t), \Omega(z)) = 0. \quad (8)$$

The proof is a straightforward extension from the result for finite-dimensional systems where $\mathcal{X} \subset \mathbb{R}^n$ (see, for example, La Salle [9] or Logemann and Ryan [10]). This result will be used for an infinite-dimensional system in Section 3. The proof is given below to make the paper self-contained. We mention that $\Omega(z)$ is also connected.

Proof: Since $z(\mathbb{R}_+)$ is relatively compact, $\Omega(z)$ is non-empty and compact.

To prove π -invariance, take $\xi \in \Omega(z)$, so that there exists a sequence (t_n) in \mathbb{R}_+ such that $t_n \rightarrow \infty$ and $z(t_n) \rightarrow \xi$. Take $t > 0$, then

$$\pi(t, \xi) = \lim_{n \rightarrow \infty} \pi(t, z(t_n)) = \lim_{n \rightarrow \infty} \pi(t + t_n, x_0) \in \Omega(z),$$

so that $\pi(t, \Omega(z)) \subset \Omega(z)$. To prove the opposite inclusion, take $\eta \in \Omega(z)$, so that $\eta = \lim_{n \rightarrow \infty} z(\tau_n)$ for some sequence (τ_n) with $\tau_n \rightarrow \infty$. The sequence $\pi(\tau_n - t, x_0)$ (defined for n large enough, so that $\tau_n - t > 0$) being contained in a compact set, has a convergent subsequence $\pi(\theta_n, x_0)$, where (θ_n) is a subsequence of $(\tau_n - t)$. If we put $\xi = \lim_{n \rightarrow \infty} \pi(\theta_n, x_0)$, then $\pi(t, \xi) = \eta$.

To prove (8), assume that (8) is false. Then there exists a sequence $(t_n) \in \mathbb{R}_+$ such that $t_n \rightarrow \infty$ and $\mu(z(t_n), \Omega(z)) \geq \varepsilon > 0$ for all n . This is a contradiction since for a subsequence (θ_n) of (t_n) , we have $z(\theta_n) \rightarrow \xi \in \Omega(z)$. \square

\mathbf{P} is said to be *zero-state detectable* if the following is true: If $u(t) = 0$ and x is a solution of (4) on $[0, \infty)$ such that $y(t) = 0$ for all $t \geq 0$, then $\lim_{t \rightarrow \infty} x(t) = 0$.

III. MAIN RESULTS

We consider the system \mathbf{P} described by (4) and (5), with the mild assumptions on f and g stated after (5).

We need additional assumptions on the function f :

(A1) For every compact set $\mathcal{K} \subset \mathbb{R}^n$, there exist constants $c_1, c_2 > 0$ such that

$$\|f(x_1, u) - f(x_2, u)\| \leq (c_1 + c_2 \|u\|^2) \|x_1 - x_2\|, \quad (9)$$

for all $u \in \mathbb{R}^m$ and $x_1, x_2 \in \mathcal{K}$.

(A2) For each fixed $a \in \mathbb{R}^n$, there exist constants $c_3, c_4 > 0$ such that

$$\|f(a, u)\| \leq c_3 + c_4 \|u\|^2 \quad \forall u \in \mathbb{R}^m. \quad (10)$$

Remark 3.1: It can be shown that **(A1)** and **(A2)** are satisfied for affine nonlinear systems \mathbf{P} described by

$$\dot{x} = \tilde{f}(x) + g(x)u, \quad (11)$$

$$y = h(x), \quad (12)$$

where $\tilde{f} \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$, $g \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^{n \times m})$, $g(0)$ has rank m and h is as in (5). This class of systems includes also the port-controlled Hamiltonian systems [14].

For any $\tau \geq 0$, we denote by \mathbf{S}_τ^* the left-shift operator by τ , acting on $X = L^2(\mathbb{R}_+, \mathbb{R}^m)$. The reason for this notation is that, traditionally, \mathbf{S}_τ denotes the right-shift by τ on X and \mathbf{S}_τ^* is the adjoint of \mathbf{S}_τ . By denoting $d_0 = u$ and $d_t = \mathbf{S}_t^* d_0$, it follows that $d_t \in L^2(\mathbb{R}_+, \mathbb{R}^m)$ for all $t \geq 0$ and the following equation holds for almost every $t \geq 0$:

$$\frac{d}{dt} \|d_t\|_{L^2}^2 = \frac{d}{dt} \int_t^\infty \|d_0(\xi)\|^2 d\xi = -\|d_0(t)\|^2. \quad (13)$$

Theorem 3.2: Let the plant \mathbf{P} defined by (4), (5) be zero-state detectable and assume **(A1)**-(**A2**). Assume that \mathbf{P} has a storage function H such that $H(x) > 0$ for $x \neq 0$, $H(0) = 0$, H is proper and (6) (strict output passivity) holds.

Then for every initial condition $x(0) \in \mathbb{R}^n$ and for every $u \in L^2(\mathbb{R}_+, \mathbb{R}^m)$, the state trajectory x of \mathbf{P} is defined for all $t \geq 0$ and it satisfies $x(t) \rightarrow 0$ as $t \rightarrow \infty$ (and hence $y(t) \rightarrow 0$ as $t \rightarrow \infty$).

Proof: Using **(A1)**, we have that for every compact set $\mathcal{K} \subset \mathbb{R}^n$ there exist constants $c_1, c_2 > 0$ such that (9) holds. By denoting $\alpha(t) = c_1 + c_2 \|u(t)\|^2$ and since $u \in L^2(\mathbb{R}_+, \mathbb{R}^m)$, it is easy to see that α is locally integrable and satisfies the condition **(S1)** in Theorem 2.2.

Using the assumption **(A2)**, we have that for each fixed $a \in \mathbb{R}^n \times \mathbb{R}^l$, there exist constants $c_3, c_4 > 0$ such that (10) holds. By denoting $\beta(t) = c_3 + c_4 \|u(t)\|^2$ and since $u \in L^2(\mathbb{R}_+, \mathbb{R}^m)$, β is locally integrable and satisfies the condition **(S2)** in Theorem 2.2 for the state equation (4).

Then using α, β as above and using initial value $x(0) \in \mathbb{R}^n$, it follows from Theorem 2.2 that there exists $\delta > 0$ and a unique solution of (14) with input $u \in L^2(\mathbb{R}_+, \mathbb{R}^m)$ on $\mathcal{I} = [0, \delta)$. In particular, x is absolutely continuous as a function of t on \mathcal{I} .

We define an infinite-dimensional signal generator for the signal u . This signal generator has the state space $X = L^2(\mathbb{R}_+, \mathbb{R}^m)$ and the evolution of its state is governed by the operator semigroup $(\mathbf{S}_\tau^*)_{\tau \geq 0}$. Thus, the state of the signal generator at time t is $d_t = \mathbf{S}_t^* d_0$, where $d_0 \in X$ is the initial

state. The generator of this semigroup is $\mathcal{A} = \frac{d}{d\xi}$ with domain $\mathcal{D}(\mathcal{A}) = \mathcal{H}^1(\mathbb{R}_+, \mathbb{R}^m)$. The observation operator of this signal generator is \mathcal{C} , defined for $\phi \in \mathcal{D}(\mathcal{A})$ by $\mathcal{C}\phi = \phi(0)$. It can be checked that \mathcal{C} is admissible in the sense of Weiss [18]. We need the Lebesgue extension of \mathcal{C} , denoted by \mathcal{C}_L , defined by

$$\mathcal{C}_L\phi = \lim_{\varepsilon \rightarrow 0} \mathcal{C} \frac{1}{\varepsilon} \int_0^\varepsilon \mathbf{S}_t^* \phi \, dt = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^\varepsilon \phi(\xi) \, d\xi.$$

with $\mathcal{D}(\mathcal{C}_L)$ being the set of all $\phi \in X$ for which the above limit exists. We refer to [18] for more information on the concept of Lebesgue extension. The output function of the signal generator is $u(t) = \mathcal{C}_L d_t$, which is defined for almost every $t \geq 0$. It turns out that $u = d_0$ (the generated signal is the initial state).

We define an extended system \mathbf{L} by connecting \mathbf{P} to the generator for d_0 as shown in Figure 1. Then we have

$$\dot{x}(t) = f(x(t), u(t)), \quad (14)$$

$$d_t = \mathbf{S}_t^* d_0, \quad (15)$$

$$u(t) = \mathcal{C}_L d_t, \quad (16)$$

$$y(t) = h(x(t)). \quad (17)$$

Let $z(t) = \begin{bmatrix} x(t) \\ d_t \end{bmatrix}$ denote the state at time t of the above system, so that $z(t) \in Z = \mathbb{R}^n \times X$.

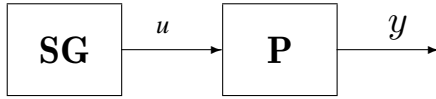


Fig. 1. The extended closed-loop system. The block \mathbf{SG} is the infinite-dimensional linear signal generator for the L^2 signal u .

Consider the storage function $H_{cl} : Z \rightarrow \mathbb{R}_+$ defined for $z = \begin{bmatrix} x \\ d \end{bmatrix}$ by $H_{cl}(z) = H(x) + \gamma \|d\|^2$ where $\gamma > \frac{1}{4k}$, where $k > 0$ is the constant from (6). We show that $H_{cl}(z(t))$ is absolutely continuous as a function of t . Since $H \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}_+)$ and the solution x of (14) is absolutely continuous as a function of t defined in \mathcal{I} , it follows that $H(x(t))$ is absolutely continuous on \mathcal{I} . From (13) and since $d_0 \in L^2(\mathbb{R}_+, \mathbb{R}^m)$, it follows that $\frac{d}{dt} \|d_t\|_{L^2}^2 \in L^1(\mathbb{R}_+, \mathbb{R}^m)$. This implies that $\|d_t\|_{L^2}^2$ is absolutely continuous on \mathbb{R}_+ .

Using (6), (13), (14) – (16), we obtain that, for almost every $t \in \mathcal{I}$,

$$\begin{aligned} \dot{H}_{cl} &= \frac{\partial H(x)}{\partial x} f(x, d_0(t)) - \gamma \|d_0(t)\|^2 \\ &\leq \langle y, d_0(t) \rangle - k \|y\|^2 - \gamma \|d_0(t)\|^2, \\ &\leq \left(\frac{1}{2\theta} - \gamma \right) \|d_0(t)\|^2 + \left(\frac{\theta}{2} - k \right) \|y\|^2 \quad \forall \theta > 0. \end{aligned}$$

By choosing $\theta \in (1/2\gamma, 2k)$, we obtain

$$\dot{H}_{cl}(z(t)) \leq -c_5 \|u(t)\|^2 - c_6 \|y(t)\|^2 \leq 0, \quad (18)$$

where $c_5 = \gamma - \frac{1}{2\theta} > 0$ and $c_6 = k - \frac{\theta}{2} > 0$.

Let us prove that $\mathcal{I} = \mathbb{R}_+$. If the maximal interval of definition of a state trajectory is $\mathcal{I} = [0, \delta)$ with $\delta < \infty$,

then it follows from Corollary 2.3 that $x(t)$ must leave any compact set $\mathcal{K} \subset \mathbb{R}^n$ at some finite time $T < \delta$. Since H_{cl} is absolutely continuous as a function of t and bounded from below, (18) implies that $H_{cl}(z(t))$ is bounded and non-increasing for all $t \in \mathcal{I}$. In particular, the state $x(t)$ never leaves the compact set $\{x \in \mathbb{R}^n \mid H(x) \leq H_{cl}(z(0))\}$ for all $t \in \mathcal{I}$. This contradiction shows that $\mathcal{I} = \mathbb{R}_+$ and $H_{cl}(z(t))$ has a limit h as $t \rightarrow \infty$.

We will prove the relative compactness of $z(\mathbb{R}_+)$. It has been shown that $x(t)$ is bounded for all $t \in \mathbb{R}_+$, hence $x(\mathbb{R}_+)$ is relatively compact in \mathbb{R}^n . Since $\lim_{t \rightarrow \infty} \|d_t\|_{L^2} = 0$, the mapping $t \mapsto d_t$ is a continuous mapping from the compact interval $[0, \infty]$ to $L^2(\mathbb{R}_+, \mathbb{R}^m)$. (Here, $[0, \infty]$ is the compactification of \mathbb{R}_+ .) The image of a compact set through a continuous mapping is always compact. Thus, the state trajectory of the signal generator together with its limit point 0 is a compact set in $L^2(\mathbb{R}_+, \mathbb{R}^m)$, i.e., the set $\{d_t \mid t \geq 0\}$ is relatively compact in $L^2(\mathbb{R}_+, \mathbb{R}^m)$. Therefore $z(\mathbb{R}_+)$ is relatively compact in $\mathbb{R}^n \times X$.

Let π denote the semiflow of (14)–(15) so that $z(t) = \pi(t, z_0)$. According to Proposition 2.4 and the relative compactness of $z(\mathbb{R}_+)$, $\Omega(z)$ is non-empty, compact and π -invariant.

For any $\xi \in \Omega(z)$, there is a sequence (t_n) in \mathbb{R}_+ such that $t_n \rightarrow \infty$ and $z(t_n) \rightarrow \xi$. By the continuity of H_{cl} , $H_{cl}(\xi) = \lim_{n \rightarrow \infty} H_{cl}(z(t_n)) = h$. Therefore, $H_{cl}(z(t)) = h$ on $\Omega(z)$. Since $\Omega(z)$ is π -invariant, $\Omega(z) \subset E = \{z \mid \dot{H}_{cl}(z) = 0\}$.

Let M be the largest π -invariant set contained in E . Since $\Omega(z)$ is π -invariant and $\Omega(z) \subset E$, we have $\Omega(z) \subset M$.

In the invariant set M , H_{cl} is constant along state trajectories and $y = 0$ and $u = 0$ along such trajectories. By the assumptions of the theorem, \mathbf{P} is zero-state detectable, i.e., if $u(t) = 0$ and $y(t) = 0$ for all $t \in \mathbb{R}_+$ then $x(t) \rightarrow 0$ as $t \rightarrow \infty$. Also, if $u(t) = 0$ for all $t \in \mathbb{R}_+$ then $d_0 = 0$, so that $d_t = 0$ for all $t \in \mathbb{R}_+$. Hence, in the invariant set M , $H_{cl}(z) = H_{cl}(0) = 0$ for all $z \in M$. Since $H_{cl}(z) > 0$ for all $z \neq 0$, we obtain that $M = \{0\}$, hence $\Omega(z) = \{0\}$. Using (8) it follows that $x(t) \rightarrow 0$ as $t \rightarrow \infty$.

The above argument is valid for any initial state $x(0) \in \mathbb{R}^n$ and for any $u \in L^2(\mathbb{R}_+, \mathbb{R}^m)$. \square

Corollary 3.3: Let the plant \mathbf{P} be as in Theorem 3.2. Then for every $x(0) \in \mathbb{R}^n$ there exists a unique solution of (4) with $u \in L_{loc}^2(\mathbb{R}_+, \mathbb{R}^m)$ in \mathbb{R}_+ .

Proof: To prove the result, we use a contradiction. Suppose that there exists an input $u \in L_{loc}^2(\mathbb{R}_+, \mathbb{R}^m)$ and a finite escape time $T > 0$ for the trajectory of x of the system with initial conditions $x(0) = x_0$. According to Corollary 2.3, $\|x(t)\| \rightarrow \infty$ as $t \rightarrow T$. Then using \tilde{u} given by

$$\tilde{u}(t) = \begin{cases} u(t) & \forall t \in [0, T], \\ 0 & \forall t \in (T, \infty), \end{cases}$$

the trajectory \tilde{x} of the system with $\tilde{x}(0) = x_0$ and input \tilde{u} also has the same finite escape time T . This is a contradiction. Indeed, since $\tilde{u} \in L^2(\mathbb{R}_+, \mathbb{R}^m)$, it follows from Theorem 3.2

that the state trajectory \tilde{x} corresponding to \tilde{u} is bounded for $t \in [0, \infty)$, i.e., there is no finite escape time. \square

Note that the convergence of the state trajectory x to zero does not imply that $x \in L^2(\mathbb{R}_+, \mathbb{R}^n)$. We give an example where $u \in L^2(\mathbb{R}_+, \mathbb{R}^m) \Rightarrow y \in L^2(\mathbb{R}_+, \mathbb{R}^m)$ with a unique solution of the state $x(t)$ for all $t \in \mathbb{R}_+$, but $x \notin L^2(\mathbb{R}_+, \mathbb{R}^n)$. Let the strictly output passive plant \mathbf{P} be described by

$$\dot{x} = -x^3 + u, \quad y = x^3, \quad (19)$$

where $x(t), u(t), y(t) \in \mathbb{R}$. Using the storage function $H(x) = \frac{1}{4}x^4$, it follows from Theorem 3.2 that for every $u \in L^2(\mathbb{R}_+, \mathbb{R})$ and every initial state $x(0) \in \mathbb{R}$, there exists a unique solution $x(t)$ of (19) in \mathbb{R}_+ and $\lim_{t \rightarrow \infty} |x(t)| = 0$.

However, this does not imply that $x \in L^2(\mathbb{R}_+, \mathbb{R})$. Using $u = 0$ and initial state $x(0) = a$, the solution x of (19) is given by

$$x(t) = \left(2t + \frac{1}{a^2}\right)^{-0.5},$$

so that $x \notin L^2(\mathbb{R}_+, \mathbb{R})$.

IV. SYSTEM STABILITY

Consider the following single-input single-output plant \mathbf{P}

$$\dot{x} = -xu^{2p} - x + u, \quad y = x, \quad (20)$$

where p is a positive integer. This plant \mathbf{P} is strictly output passive. Indeed, using the storage function $H(x) = \frac{1}{2}x^2$, we have

$$\begin{aligned} \dot{H} &= -x^2u^{2p} - x^2 + xu \\ &\leq \langle y, u \rangle - \|y\|^2. \end{aligned}$$

From this inequality, it can be shown that \mathbf{P} has a finite L^2 gain of 1, i.e., $\|y_T\|_{L^2} \leq \|u_T\|_{L^2} + \sqrt{2H(x(0))}$ (see Lemma 6.5 in [8] for details).

However, this does not imply that for every $u \in L^2(\mathbb{R}_+, \mathbb{R})$ the solution $x(t)$ of (20) exists on some interval $t \in [0, \delta)$ with $\delta > 0$. Indeed, consider $p \geq 2$ and

$$u(t) = \begin{cases} t^{-\frac{1}{2p}} & \text{for } t \in [0, 1), \\ 0 & \text{for } t \in [1, \infty), \end{cases} \quad (21)$$

so that $u \in L^2(\mathbb{R}_+)$. Now the state equation (20) can be written as follows:

$$\dot{x} = -xt^{-1} - x + t^{-\frac{1}{2p}} \quad \forall t \in [0, 1). \quad (22)$$

It can be shown that if $x(0) \neq 0$ then a solution of (22) does not exist on any interval $[0, \delta)$, where $\delta > 0$. Without loss of generality, assume that $x(0) < 0$. Using contradiction, suppose that there exists a solution x of (22) on $[0, \delta)$ with $\delta > 0$. By the continuity of x on $[0, \delta)$, there exists $\varepsilon \in [0, \delta)$ such that $x_\varepsilon = \max_{t \in [0, \varepsilon)} x(t) < 0$. By Definition 2.1, the state trajectory x satisfies

$$\begin{aligned} x(t) &= x(0) + \int_0^t [-(\tau^{-1} + 1)x(\tau) + \tau^{-\frac{1}{2p}}] d\tau \\ &> x(0) + \int_0^t -(\tau^{-1} + 1)x_\varepsilon d\tau + \int_0^t \tau^{-\frac{1}{2p}} d\tau \\ &= \infty \end{aligned}$$

for all $t \in (0, \varepsilon)$. This contradicts the existence of a solution x on $[0, \delta)$.

Note that if $x(0) = 0$, then the solution of (22) exists on \mathbb{R}_+ and for $t \in (0, 1]$ it is given by

$$x(t) = e^{(-\ln(t)-t)} \int_0^t e^{(\ln(\tau)+\tau)} \tau^{-\frac{1}{2p}} d\tau, \quad (23)$$

(this can be verified directly).

It has been shown that the plant \mathbf{P} as in (20) with input u as in (21) does not have a solution on any interval of the type $[0, \delta)$ when $p \geq 2$. It has a unique solution when $p = 1$ which can also be concluded from Theorem 3.2 since it satisfies Assumption (A1).

Now consider plant \mathbf{P} described by

$$\dot{x} = \begin{cases} -x + \text{sat}(u) & \forall x \in [-1, 1], \\ x - 2 + \text{sat}(u) & \forall x \in (1, \infty), \\ x + 2 + \text{sat}(u) & \forall x \in (-\infty, -1), \end{cases} \quad (24)$$

$$y = x, \quad (25)$$

where $x(t), u(t), y(t) \in \mathbb{R}$, $\text{sat} : \mathbb{R} \rightarrow \mathbb{R}$ is a saturation function defined by $\text{sat}(u) = u$ for all $u \in (-1, 1)$ and $\text{sat}(u) = u/|u|$ otherwise. For the plant \mathbf{P} as in (24), (25) and for every initial condition $x(0) \in \mathbf{B}_1$, it can be checked that every $u \in L^2(\mathbb{R}_+, \mathbb{R})$ implies the existence of a unique solution and the corresponding output $y \in L^2(\mathbb{R}_+, \mathbb{R})$. But when $x(0)$ is outside the ball \mathbf{B}_3 , i.e., $x(0) \in \mathbb{R} \setminus \mathbf{B}_3$, $u \in L^2 \nRightarrow y \in L^2$ and $\lim_{t \rightarrow \infty} \|y(t)\| = \infty$.

The concept of L^2 -stability is originally defined for mapping, see, for example, Vidyasagar [17, Chapter 6.3] or van der Schaft [14, Chapter 1.2]. Its generalization to state equations often overlooks the influence of the initial state on the output (for example in [17, Chapter 6.3]) or the existence of solution of the state equation (for example in [14, Remark 3.1.4] or in [8, Lemma 6.5]). Example (20) with $p \geq 2$ shows that the system having a finite L^2 -gain (in the sense of [14, Definition 3.1.3]) does not imply L^2 -stability. Example (24) shows that for every initial condition $x(0)$ in a compact set, every L^2 input u implies the existence of a unique solution to the system equations and the corresponding output y is in L^2 , but this property does not hold anymore when the initial condition $x(0)$ is outside the set.

A good definition of L^2 -stability for state equations is given in [14, Definition 1.2.11] but it omits the boundedness of the state trajectories. This omission allows an LTI system to be categorized as an L^2 -stable system (in the sense of [14, Definition 1.2.11]) but the state grows unbounded for any L^2 input, for example, the plant \mathbf{P} given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \quad (26)$$

In this section, we want to refine again the concept of L^2 -stability for dynamical systems which combines the L^2 -stability concept from van der Schaft [14] or Vidyasagar [17] with the concept of system stability for linear systems as defined in Curtain [2].

Definition 4.1: The plant \mathbf{P} described by (4) is L^2 system-stable if for every $u \in L^2(\mathbb{R}_+, \mathbb{R}^m)$ and $x(0) \in \mathbb{R}^n$, there

exists a unique solution x of (4) on \mathbb{R}_+ , the state trajectory x is bounded and the output function y is in $L^2(\mathbb{R}_+, \mathbb{R}^m)$.

It follows that any plant \mathbf{P} satisfying the assumptions in Theorem 3.2 is L^2 system-stable, while the plant \mathbf{P} in (20) with $p \geq 2$, the plant \mathbf{P} in (24),(25) and the plant \mathbf{P} in (26) are not L^2 system-stable. Note that if a plant \mathbf{P} is L^2 system-stable then it is also L^2 -stable.

Proposition 4.2: Let the plant \mathbf{P} be defined by (4) and assume **(A1)**–**(A2)**. Assume that \mathbf{P} has a storage function H such that $H(x) > 0$ for $x \neq 0$, $H(0) = 0$, H is proper and \mathbf{P} is *strictly output passive*, i.e.,

$$\dot{H} \leq \langle y, u \rangle - k\|y\|^2 \quad (27)$$

holds with $k > 0$. Then \mathbf{P} is L^2 system-stable.

Proof: Let $u \in L^2(\mathbb{R}_+, \mathbb{R}^m)$. It follows from the first part of the proof in Theorem 3.2 that for any initial conditions $x(0) \in \mathbb{R}^n$ that there exists a global solution x of (4) and the state trajectory x is bounded.

By the strict output passivity of \mathbf{P} , we have

$$\|y\|_{L^2} \leq \frac{1}{k}\|u\|_{L^2} + \sqrt{\frac{2}{k}H(x(0))}.$$

Thus $y \in L^2(\mathbb{R}_+, \mathbb{R}^m)$. \square

Corollary 4.3: Let the plant \mathbf{P} be as in Proposition 4.2. Then for every $x(0) \in \mathbb{R}^n$ and for every $u \in L^2_{loc}(\mathbb{R}_+, \mathbb{R}^m)$ there exists a global unique solution of (4).

Remark 4.4: A passive system with a proper storage function and satisfying **(S3)**–**(S4)**, does not necessarily have a global solution for every input $u \in L^2(\mathbb{R}_+)$. Indeed, let the plant \mathbf{P} be given by

$$\dot{x} = (1-x)^2 u \quad y = x(1-x)^2,$$

where $x(t), u(t), y(t) \in \mathbb{R}$, with the proper storage function $H = \frac{1}{2}x^2$. \mathbf{P} is passive, since $\dot{H} = \langle y, u \rangle$. Note that \mathbf{P} satisfies **(A1)**–**(A2)** but it is not strictly output passive. Suppose that the input u is given by

$$u(t) = \begin{cases} -2 & \forall t \in [0, 1) \\ 0 & \text{elsewhere,} \end{cases}$$

so that $u \in L^2(\mathbb{R}_+)$ and consider the initial condition $x(0) = 0.5$. Then the solution of the differential equation is $x(t) = 1 - (2 - 2t)^{-2}$, which is defined only on $[0, 1)$ and $\lim_{t \rightarrow 1^-} x(t) = -\infty$. \square

V. SYSTEM-STABLE INTERCONNECTIONS

The motivation to study L^2 system-stability is analogous to the study of Input-to-State Stability (ISS). By definition, for an ISS system with input u and state x , any input $u \in L^\infty$ implies that there exists a global solution x of the state equation and $x \in L^\infty$. If we define an output y which depends continuously on the state x , then it follows that $u \in L^\infty \Rightarrow y \in L^\infty$. In the same manner, an L^2 system-stable with input u , state x and output y has the property that any input $u \in$

L^2 implies that there exists a global solution x of the state equation, $x \in L^\infty$ and $y \in L^2$.

A cascade connection of two ISS retains the ISS property of the interconnected systems. The same consequence also applies to the cascade connection of two plants which are L^2 system-stable. Let the plants \mathbf{P}_i , $i = 1, 2$, be given by

$$\dot{x}_i = f_i(x_i, u_i), \quad y_i = h_i(x_i) \quad (28)$$

where $x_i(t) \in \mathbb{R}^{n_i}$ and $u_i(t), y_i(t) \in \mathbb{R}^{m_i}$. Consider $m_1 = m_2$ and $\mathbf{P}_1, \mathbf{P}_2$ are L^2 system-stable and are cascade connected by $u_2 = y_1$. Then the whole system with input u_1 , state $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and output y_2 is L^2 system-stable. Indeed, by L^2 system-stability of \mathbf{P}_1 , any $u_1 \in L^2$ implies the global solution of x_1 , and we have $x_1 \in L^\infty$ and $y_1 \in L^2$. Since $u_2 = y_1 \in L^2$, by L^2 system-stability of \mathbf{P}_2 , there exists global solution of x_2 , and we have $x_2 \in L^\infty$ and $y_2 \in L^2$.

The feedback interconnection of ISS systems preserves the ISS property of the closed loop system provided that a small-gain type condition is satisfied (see Jiang *et al* [7] for details). The feedback interconnection version for L^2 system-stable is given in the following proposition.

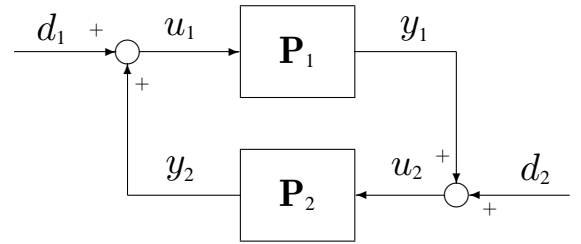


Fig. 2. The feedback interconnection of systems stable \mathbf{P}_1 and \mathbf{P}_2 .

Proposition 5.1: Let the plants \mathbf{P}_i , $i = 1, 2$, be given by (28) with $m_1 = m_2$. Suppose that for each $i = 1, 2$, f_i assumes **(A1)**–**(A2)** and \mathbf{P}_i is L^2 system-stable. Assume that for each $i = 1, 2$, \mathbf{P}_i has a finite L^2 -gain denoted by γ_i . Suppose that \mathbf{P}_1 and \mathbf{P}_2 are feedback interconnected as in Figure 2 such that $u_1 = d_1 + y_2$ and $u_2 = d_2 + y_1$ where d_1, d_2 are external signals. If $\gamma_1 \gamma_2 < 1$ then the closed-loop system with input $\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$ and output $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ is L^2 system-stable.

Proof: Let $\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \in L^2(\mathbb{R}_+, \mathbb{R}^{m_1+m_2})$. The closed-loop system \mathbf{L} is given by the following state equation

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, h_2(x_2) + d_1) \\ \dot{x}_2 &= f_2(x_2, h_1(x_1) + d_2). \end{aligned} \quad (29)$$

Using **(A1)**, we have that for every compact set $\mathcal{B} \subset \mathbb{R}^{n_1+n_2}$ there exist constants $c_1, c_2 > 0$ such that (9) holds for the closed-loop system \mathbf{L} . By denoting $\alpha(t) = c_1 + c_2 \left\| \begin{bmatrix} d_1(t) \\ d_2(t) \end{bmatrix} \right\|^2$ and since $\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \in L^2(\mathbb{R}_+, \mathbb{R}^{m_1+m_2})$, it is easy to see that α is locally integrable and satisfies the condition **(S1)** in Theorem 2.2.

Using the assumption **(A2)**, we have that for each fixed $a \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, there exist constants $c_3, c_4 > 0$ such that (10) holds for \mathbf{L} . By denoting $\beta(t) = c_3 + c_4 \left\| \begin{bmatrix} d_1(t) \\ d_2(t) \end{bmatrix} \right\|^2$ and since

$\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \in L^2(\mathbb{R}_+, \mathbb{R}^{m_1+m_2})$, β is locally integrable and satisfies the condition (S2) in Theorem 2.2 for the state equation (29).

It follows from Theorem 2.2 (with α, β as above) that for any initial value $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, there exists a maximal interval time of definition $\delta > 0$ and a unique solution of (29) with input $\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \in L^2(\mathbb{R}_+, \mathbb{R}^m)$ on $\mathcal{J} = [0, \delta)$. In particular, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is absolutely continuous on \mathcal{J} .

For any measurable function f defined on \mathcal{J} , we denote by $\|f\|_{L^2(\mathcal{J})} = (\int_0^\delta \|f(t)\|^2 dt)^{\frac{1}{2}}$. Using the finite L^2 gain of \mathbf{P}_1 and \mathbf{P}_2 in the interval time of definition \mathcal{J} , we have

$$\begin{aligned} \|y_1\|_{L^2(\mathcal{J})} &\leq \gamma_1 \|d_1 + y_2\|_{L^2(\mathcal{J})} + \beta_1 \\ \|y_2\|_{L^2(\mathcal{J})} &\leq \gamma_2 \|d_2 + y_1\|_{L^2(\mathcal{J})} + \beta_2 \end{aligned}$$

where $\beta_1, \beta_2 \in \mathbb{R}$. By simple algebraic manipulation, it can be shown that

$$\begin{aligned} \|y_1\|_{L^2(\mathcal{J})} &\leq \frac{1}{1 - \gamma_1 \gamma_2} \left(\gamma_1 \|d_1\|_{L^2(\mathcal{J})} + \gamma_1 \gamma_2 \|d_2\|_{L^2(\mathcal{J})} \right. \\ &\quad \left. + \beta_1 + \gamma_1 \beta_2 \right) \\ \|y_2\|_{L^2(\mathcal{J})} &\leq \frac{1}{1 - \gamma_1 \gamma_2} \left(\gamma_1 \gamma_2 \|d_1\|_{L^2(\mathcal{J})} + \gamma_2 \|d_2\|_{L^2(\mathcal{J})} \right. \\ &\quad \left. + \gamma_2 \beta_1 + \beta_2 \right). \end{aligned}$$

Since $\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \in L^2(\mathbb{R}_+, \mathbb{R}^{m_1+m_2})$, it implies that $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in L^2(\mathcal{J}, \mathbb{R}^{m_1+m_2})$. It follows also that $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in L^2(\mathcal{J}, \mathbb{R}^{m_1+m_2})$. By the L^2 system-stability of \mathbf{P}_1 and \mathbf{P}_2 , x_1 and x_2 is bounded on \mathcal{J} . Using Corollary 2.3 we conclude that the maximal interval of definition of $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is \mathbb{R}_+ . Hence the state trajectory $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is bounded on \mathbb{R}_+ and $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in L^2(\mathbb{R}_+, \mathbb{R}^{m_1+m_2})$. \square

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